International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 6, Issue 6, Oct – Nov 2017; 37-50

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

PREDICTION OF THE POPULATION OF BANGLADESH, USING LOGISTIC MODEL

MD. ASRAFUL ISLAM & PAYER AHMED

Department of Mathematics, Jagannath University, Dhaka, Bangladesh

ABSTRACT

In this paper, we have discussed logistic mathematical model, to analyze the population problems of Bangladesh. We have also discussed analytic solution, geometrical analysis, stability and local linearization of the logistic growth model. We analyzed the logistic model, both analytically and numerically. Several census data are analyzed for some countries, including Bangladesh. We have analyzed the population growth of Bangladesh, for a certain period of time (1971-2011). Finally, we implement the logistic model, that gives the future population of Bangladesh during (2012-2050).

KEYWORDS: Analytic Solution, Carrying Capacity, Equilibrium Point, Geometrical Analysis, Logistic Population Model, Local Linearization, Stability

1.0 INTRODUCTION

The logistic model was named in 1844-1845, by Pierre Francois Verhulst, who studied it in relation to population growth [1]. Verhulst derived his logistic equation, to describe the self-limiting growth of a biological population. The equation was rediscovered in 1911 by A.G. Mckendrick, for the growth of bacteria in both and experimentally tested using a technique, for nonlinear parameter estimates [2]. The model is also sometimes called Pearl, R., & Reed, L. J. (1920), on the rate of growth of the population of the United States, since 1790 [3]. The carrying capacity $k(t)$ leads to a logistic delay equation, which has a very rich behavior[4]. The main point about the logistic model is a particularly convenient form to take, when seeking qualitative dynamic behavior in populations in which $N=0$ is an unstable steady state and $N(t)$ tends to a finite positive stable steady state[5]. The logistic form will occur in a variety of different contexts, throughout the paper primarily, because of its algebraic simplicity and because, it provides a preliminary qualitative idea of what can occur, with more realistic forms. It is instructive to try to understand, why the logistic form was accepted since, it highlights an important point in modeling in the biomedical sciences. The logistic growth form in this paper contains three parameters, N_0, k and r with which, to assign to compare with actual data. These were used by Pearl (1925), to fit the census population data, for various countries, including the United States and France for various periods. The main point is not that, the predictions are so inaccurate, but rather that curve fitting, only of the data, and particularly the part which does not cover the major part of the growth curve makes comparison, with data and future predictions extremely unreliable. The population problem is one of the main problems in Bangladesh at the current time. Bangladesh is an overpopulated country and the growth in resources has not been keeping pace with the growth in population. So, increasing the trend in population is a great threat to the nation. Kabir and Chowdhury investigated the relationship between population growth and food production in Bangladesh [6]. Recognizing the difficulty of feeding the growing population, even with considerable increase in food production, they suggested giving priority to population policy, for reduction in population. In this situation, prediction of population is very essential for planning.

2.0 Formulation of the Logistic Model

Considering the fact that the growth rate actually depends on the population, we replace the constant r in $\frac{dN}{dt} = rN$ by a function $h(N)$ and thus, we obtained the modified equation $\frac{dN}{dt} = h(N)N$ **(2.1)**

We now choose $h(N)$ so that $h(N) \cong r > 0$ when N is small, $h(N)$ decreases as N grows larger, and $h(N) < 0$ when N is sufficiently large. Having this property, the simplest function is $h(N) = r - aN$, where a is a positive constant. This (2.1) becomes $\frac{dN}{dt} = (r - aN)N$ **(2.2)**

which is known as **Verhulst equation** or the **Logistic equation**. Its solution is called the logistic function. The graph of a logistic function is called a logistic curve.

[P.F. Verhulst (1804-1849) was a Belgian mathematician who introduced an equation (2.2) as a model for human population growth in 1838. He referred to it as a logistic growth; hence equation (2.2) is often called the logistic equation. Due to inadequate census data he was unable to test the accuracy of his model.]

Suppose an environment is capable of sustaining no more than a fixed number k of individuals in its population, the quantity is called the carrying capacity of the environment.

Hence the function h in (2.1) we have $h(k) = 0$, and we simply let $h(0) = r$. The simplest assumption that we can make is that $h(N)$ is linear, that is, $h(N) = c_1 N + c_2$. Then $h(0) = r$ gives $c_1.0 + c_2 = r \Rightarrow c_2 = r$.

And $h(k) = 0$ gives $c_1 k + c_2 = 0 \Rightarrow \frac{-c_2}{k} = \frac{-r}{k}$

$$\therefore \; h(N) = \frac{-r}{k} N + r = r(1 - \frac{N}{K}) \quad \textbf{(2.3)}$$

The equivalent form of the logistic equation is $\frac{dN}{dt} = r(1 - \frac{N}{K})N$

Where $k = \frac{r}{a}$. The constant r is called the intrinsic growth rate, that is the growth rate in the absence of any limiting point.

3.0 Carrying Capacity

The carrying capacity of a biological species in an environment is the maximum population size of the species that the environment can sustain indefinitely, given food, habitat, water, and other necessities available in the environment.

3.1 Analytic Solution of Logistic Model

Verhulst suggested the following model.

$$\frac{dN}{dt} = r(1 - \frac{N}{K})N \quad \textbf{(3.1)}$$

Where $N = N(t) =$ population of a single species at time t, r and k are positive constants.

Here $r(1-\frac{N}{k})$ is the per capita birth rate. The constant k is the carrying capacity of the environment. From (2.3) we have

$$\frac{dN}{N(1-\frac{N}{k})} = rdt$$

$$\Rightarrow \frac{kdN}{N(k-N)} = rdt$$

$$\Rightarrow \left(\frac{1}{N}+\frac{1}{k-N}\right)dN = rdt$$

$\Rightarrow \int \frac{1}{N}dN + \int \frac{1}{k-N}dN = \int rdt$; [By integrating]

$\Rightarrow \ln N - \ln(k-N) + \ln A = rt; \ln A$ is integrating constant.

$$\Rightarrow \ln \frac{AN}{k-N} = rt$$

$$\Rightarrow \frac{AN}{K-N} = e^{rt}$$

$$\Rightarrow AN = (k-N)e^{rt} \quad \textbf{(3.2)}$$

Initially at $t = 0$ if $N = N(0) = N_0$ then from (3.2) we get,

$$AN_0 = (k-N_0)e^0$$

$$\Rightarrow A = \frac{k-N_0}{N_0}$$

By putting this value in (3.2) we get,

$$\frac{k-N_0}{N_0}N = (K-N)e^{rt}$$

$$\Rightarrow \frac{kN}{N_0} - N + Ne^{rt} = ke^{rt}$$

$$\Rightarrow kN - NN_0 + NN_0e^{rt} = kN_0e^{rt}$$

$$\Rightarrow N[k+N_0(e^{rt}-1)] = kN_0e^{rt}$$

$$\Rightarrow N = \frac{kN_0e^{rt}}{k+N_0(e^{rt}-1)}$$

$$N(t) = \frac{kN_0e^{rt}}{k+N_0(e^{rt}-1)} \quad \textbf{(3.3)}$$

Corollary-1: *When* $t \to \infty$ *then* $N(t) \to k$

Proof **(Corollary-1):** From the equation (3.3) we have

$$N(t) = \frac{kN_0 e^{rt}}{k + N_0(e^{rt} - 1)}$$

$$\Rightarrow N(t) = \frac{kN_0}{ke^{-rt} + N_0(1 - e^{-rt})}$$

When $t \to \infty$ then $N(t) \to \frac{kN_0}{k.0 + N_0(1-0)}$

$$\Rightarrow N(t) \to \frac{kN_0}{N_0}$$

$$\Rightarrow N(t) \to k$$

That is, we have shown that when time is unlimited, then the population approaches to the maximum, the equilibrium number of organisms of a particular species that can be supported indefinitely in a given environment.

Corollary-2: *Assume that* $t_0 < t_1 < t_2$ *are equally spaced time values. Suppose the corresponding population size are* N_0, N_1 *and* N_2 *respectively, than the carrying capacity* k *of the population that obeys the logistic law of population can be expressed as* $K = \frac{N_0 N_1^2 + N_1^2 N_2 - 2N_0 N_1 N_2}{N_1^2 - N_0 N_2}$

Proof **(Corollary-2):** From the equation (3.3) we have $AN = (k - N)e^{rt}$ **(3.4)**

Initially $t = t_0, t_1, t_2$ and $N = N_0, N_1, N_2$ respectively. So we have from (3.4)

$$AN_0 = (k - N_0)e^{rt_0} \qquad \textbf{(3.5)}$$

$$AN_1 = (k - N_1)e^{rt_1} \qquad \textbf{(3.6)}$$

$$AN_2 = (k - N_2)e^{rt_2} \qquad \textbf{(3.7)}$$

(3.5) ÷ (3.6) gives $\frac{N_0}{N_1} = \frac{k - N_0}{k - N_1} e^{r(t_0 - t_1)}$ **(3.8)**

(3.6) ÷ (3.7) gives $\frac{N_1}{N_2} = \frac{k - N_1}{k - N_2} e^{r(t_1 - t_2)}$ **(3.9)**

Since t_0, t_1, t_2 are equally spaced, so we have $t_0 - t_1 = t_1 - t_2$

Thus, (3.9) becomes $\frac{N_0}{N_1} = \frac{k - N_0}{k - N_1} e^{r(t_0 - t_1)}$ **(3.10)**

(3.8) ÷ (3.9) gives, $\frac{N_0}{N_1} \cdot \frac{N_2}{N_1} = \frac{k - N_0}{k - N_1} \cdot \frac{k - N_2}{k - N_1}$

$$\Rightarrow \frac{N_0 N_2}{N_1{}^2} = \frac{(k - N_0)(k - N_2)}{(k - N_1)^2}$$

$$\Rightarrow N_0 N_2 (k - N_1)^2 = N_1{}^2 (k - N_0)(k - N_2)$$

$$\Rightarrow (k_2 - 2kN_1 + N_1{}^2) N_0 N_2 = N_1{}^2 (k^2 - kN_2 - kN_0 + N_0 N_2)$$

$$\Rightarrow k^2 N_0 N_2 - 2k N_0 N_1 N_2 + N_1{}^2 N_0 N_2 = k^2 N_1{}^2 - kN_1{}^2 N_2 - kN_0 N_1{}^2 + N_0 N_1{}^2 N_2$$

$$\Rightarrow k^2 (N_0 N_2 - N_1{}^2) = k(2N_0 N_1 N_2 - N_1{}^2 N_2 - N_0 N_1{}^2)$$

$$\Rightarrow k(N_0 N_2 - N_1{}^2) = 2N_0 N_1 N_2 - N_1{}^2 N_2 - N_0 N_1{}^2 \quad [\because k \neq 0]$$

$$K = \frac{N_0 N_1^2 + N_1^2 N_2 - 2N_0 N_1 N_2}{N_1^2 - N_0 N_2}$$ (Proved). **(3.11)**

Corollary-3: *Suppose* $0 < N_0 < \frac{k}{2}$, *then* $r = \frac{1}{t} \ln \frac{\left(\frac{1}{N_0} - \frac{1}{k}\right)}{\left(\frac{1}{N_1} - \frac{1}{k}\right)}$

Proof **(Corollary-3):** From the equation (3.3) we have $N(t) = \frac{kN_0 e^{rt}}{k + N_0 (e^{rt} - 1)}$

At time t, let $N(t) = 2N_0$

$$N(t) = \frac{kN_0 e^{rt}}{k + N_0 (e^{rt} - 1)} = 2N_0$$

$$\Rightarrow ke^{rt} = 2k + 2N_0 e^{rt} - 2N_0$$

$$\Rightarrow (K - 2N_0) e^{rt} = 2(k - N_0)$$

$$\Rightarrow e^{rt} = \frac{2(k - N_0)}{k - 2N_0}$$

$$\Rightarrow rt = \ln \frac{2(k - N_0)}{k - 2N_0}$$

$$\Rightarrow t = \frac{1}{r} \ln \frac{2(k - N_0)}{k - 2N_0}$$

$$\Rightarrow r = \frac{1}{t}\ln\frac{2(k - N_0)}{k - 2N_0}$$

$$\Rightarrow r = \frac{1}{t}\ln\frac{2(k - N_0)}{2\left(\frac{k}{2} - N_0\right)}$$

$$\Rightarrow r = \frac{1}{t}\ln\frac{(k - N_0)}{\left(\frac{k}{2} - N_0\right)}$$

Suppose $k = \frac{1}{N_0} \Rightarrow N_0 = \frac{1}{k}$

Let us consider $2N_0 = N_1$

$$\Rightarrow r = \frac{1}{t}\ln\frac{\left(\frac{1}{N_0} - \frac{1}{k}\right)}{\left(\frac{1}{N_1} - \frac{1}{k}\right)}$$

4.0 Obtaining Equilibrium Points

We obtain the system's equilibrium points N^* by finding all values of N that satisfy $\frac{dN}{dt} = 0$:

$$\frac{dN}{dt} = 0 \;\Rightarrow rN^*\left(1 - \frac{N^*}{K}\right) = 0 \quad (4.1)$$

$$\therefore rN^* = 0 \;\; or \left(1 - \frac{N^*}{K}\right) = 0$$

We get $N^* = 0$ (4.2)

$N^* = K$ (4.3)

Thus, the logistic equation has exactly two equilibrium points.

Table 4.1: Behavior of Logistic Growth, Described by Equation (3.1) for Different Cases of N

N	$\frac{dN}{dt}$
$N > K$	$\frac{dN}{dt} < 0$
$0 < N < K$	$\frac{dN}{dt} > 0$
$N = K$	$\frac{dN}{dt} = 0$
$N = 0$	$\frac{dN}{dt} = 0$

5.0 Geometrical Analysis

Table 4.1 illustrates the sign of $\frac{dN}{dt}$ for different values N . The trivial equilibrium point N^* is unstable, and the second equilibrium point $N^* = k$ represents the stable equilibrium, where N asymptotically approaches the carrying capacity K . In terms of the limit, we can say $\lim_{T\to\infty} N(T) = k, N(0) > 0.$ A point of inflection occurs at $N = \frac{k}{2}$ for all solutions that cross it, and we can see graphically that.owth of N is rapid until it passes the inflection point $N = \frac{k}{2}$. From there, subsequent growth slows as N asymptotically approaches k, as shown in table 4.1, if $N > k$, then $\frac{dN}{dt} < 0$, and N decreases exponentially towards k. This case should only occur when the initial condition $N(0) = N_0 > k$. In the following section, we will confirm the stability of equilibrium by linearization about each equilibrium solution.

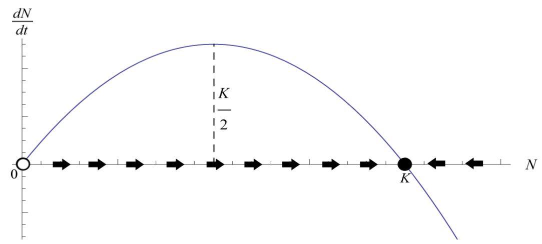

Figure 5.1: Phase Line Portrait of Logistic Growth, as Described by Equation (2.3)

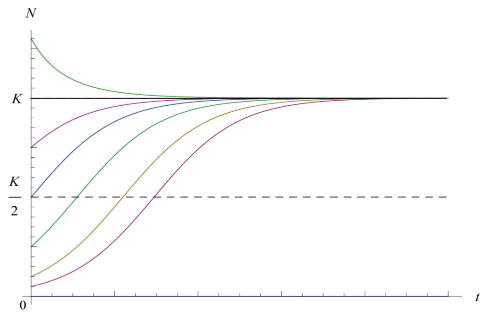

Figure 5.2: Dynamics of Logistic Growth, as Described by Equation (3.3)

6.0 Local Linearization

Let us consider a population in general to be governed by

$$\frac{dN}{dt} = f(N) \qquad \textbf{(6.1)}$$

Where *f(N)* is a continuously differentiable (typically nonlinear) function of N then the equilibrium solutions N^* are solutions of $f(N) = 0$ and linearly stable to small perturbations if $f'(N^*) < 0$, unstable if $f'(N^*) > 0$. This is clear from line arising about N^*, by writing $n(t) \approx N(t) - N^*$, $|n(t)| << 1$. Then (6.1), becomes $\frac{dN}{dt} = f(N^* + n) \approx f(N^*) + nf'(N^*) + \ldots$ which to first order in $n(t)$ gives $\frac{dN}{dt} \approx nf'(N^*) \Rightarrow n(t) \propto \exp[f'(N^*)t]$ (6.2) So n grows or decays accordingly as $f'(N^*) > 0$ or $f'(N^*) < 0$. Timescale of the response of the population to a disturbance is of the order of $\frac{1}{|f'(N^*)|}$; It is the time to change the initial disturbance by a factor e. Depending on the system $f(N)$ models, may get several equilibrium or steady state populations N^*. Graph plotting $f(N)$ against N immediately gives the equilibria as the points, where it crosses the N -axis. The gradient $f'(N^*)$ at steady state, then determines its linear stability. Such steady states may, are positive, so these equilibria are unstable are stable to small perturbations: the arrows symbolically indicate stability or instability, so that, N is in the range $N_2 < N < N_3$ then $N \to N_3$ rather than returning to N_1. A similar perturbation from N_3 to a value in the range $0 < N < N_2$ would result in $N(t) \to N_1$. Qualitatively, there is a threshold perturbation below, which the steady states are always stable, and this threshold depends on the full nonlinear form of $f(N)$. For example, the necessary threshold perturbation is $N_2 - N_1$.

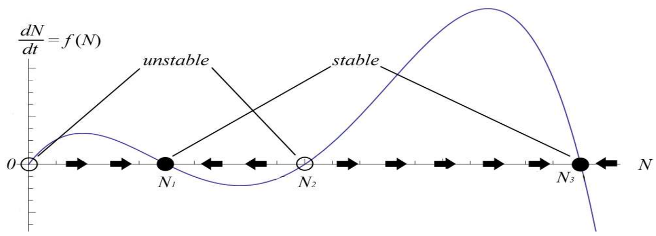

Figure 6.1: Population Dynamics Model $\frac{dN}{dt} = f(N)$ with Several Steady States. The Gradient $f'(N)$ at the Steady State. That is, where $f(N) = 0$, Determines the Linear Stability

7.0 Realistic Analysis of Logistic Model Used to Fit the Census Data

In this paper, we have discussed about the logistic model by Verhulst (1838) with its realistic situation in some countries. We have considered here the countries Belgium, USA, France and Bangladeshi populations for a certain period of times and analyzed their numerical solutions with the model. We have shown here that this model fits for either lower or upper parts of the graph-model for some countries whereas all of the lower or upper part fits for a certain country like Bangladesh and USA.

7.0 Logistic Model Used to Fit the Census Data for Belgium

Using the Corollary (2) and corollary (3) estimation of the population of Belgium in the years 1815, 1830 and 1845 (respectively 3.627, 4.247 and 4.801 million), we obtained $K = 6.584$ million and r = 2.62% per year. The population of Belgium and the logistic curve showing the following Figure (7.1).

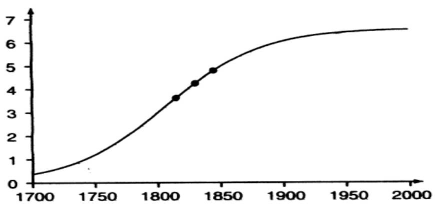

Figure 7.1: The Population of Belgium (in Million) and Logistic Curve

7.2 Logistic Model Used to Fit the Census Data for USA

Pearl (1925) used logistic model to fit the census data of USA population for various periods. Figure 7.2 shows the results for USA. If we look at the USA data there is a good fit for the population roughly from 1790 until about 1910; here the lower part of the curve is fitted and later we shown a table (Table 7.1) of USA census data and analytic value in the period (1970-2010).

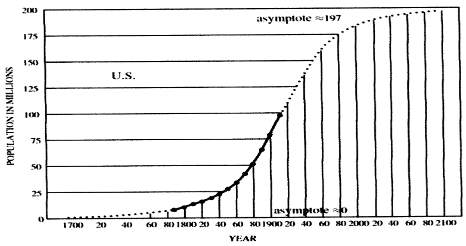

Figure 7.2: Logistic Population Growth Eq. Used to Fit the Census Data for the Population of USA (In Million)

Table 7.1: USA Census Data and Analytic Value Since (1790-2010)

Year	USA Population	Analytic
1790	3,929,214	3,929,214
1800	5,236,631	5,335,741
1810	7,239,881	7,227,100
1820	9,638,453	9,755,109
1830	12,866,020	13,106,950
1840	17,069,453	17,503,927
1850	23,191,876	23,191,876
1860	31,443,321	30,418,277
1870	38,558,371	39,391,703
1880	49,371,340	50,223,418
1890	62,979,766	62,860,165
1900	76,212,168	77,029,658
1910	92,228,531	92,228,530
1920	106,021,568	106,124,688
1930	123,202,660	121,713,981
1940	32,165,129	39,081,412
1950	151,325,798	158,280,419
1960	179,323,175	179,323,175
1970	203,211,926	202,171,349
1980	226,545,805	226,728,483
1990	248,709,873	252,835,188
2000	281,421,906	280,268,397
2010	308,745,531	308,745,531

7.3 Logistic Model Used to Fit the Census Data for France

Verhulst in 1938 did a similar study of France population, finally Pearl in 1925 used logistic model, to fit the census data of France population for various periods. Figure 7.3, shows the results for France. If we look at the France data, there is only fitted to the upper part of the curve, but even there the subsequent population growth prediction is wrong.

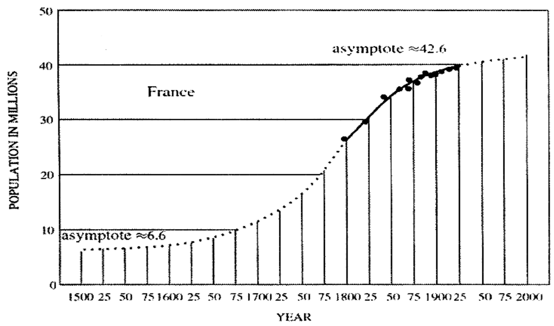

Figure 7.3: Logistic Population Growth Eq. Used to Fit the Census Data for the Population of France (In Millions)

7.4 Logistic Model Used to Fit the Census Data for Bangladesh

To fit the census data of the Bangladesh population, we need the value of carrying capacity K and growth rate r. We have used corollary (2) and corollary (3) to find carrying capacity K and growth rate r of Bangladesh population. Using Verhulst procedure that means in the help of corollary (2) and corollary (3) we show table (7.2) Bangladesh actual population, analytic and numeric data using Runge-Kutta scheme.

Table 7.2: Census, Numeric and Analytic Data of Bangladesh Population

Year	Bangladesh Population	Numeric	Analytic
1971	7.16		
1976		8.083	8.083
1981	8.98	9.069	9.069
1986		10.095	10.095
1991	11.16	11.145	12.203
2001	13.06	13.250	13.250
2006		14.271	14.272
2011	15.26	15.252	15.251

(Population in million)

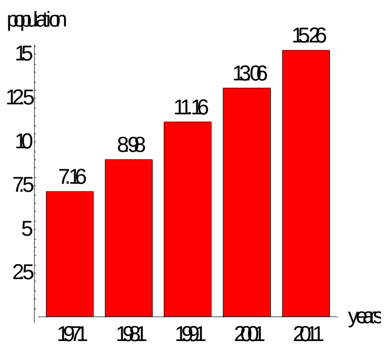

Figure 7.5: Logistic Population Growth Equation Used to Fit the Census Data for the Population of Bangladesh (In Millions)

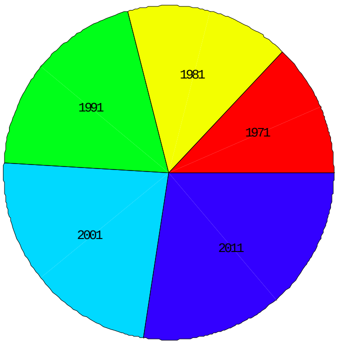

Figure 7.6: Logistic Population Growth Equation Used to Fit the Census Data for the Population of Bangladesh (In Million) by Pie Diagram

8.0 Future Population of Bangladesh Using Logistic Model

Using logistic model (2.3) we show table (8.1) Bangladesh population data from 2012 to 2050 is increasing without bound.

Table 8.1: For the population of Bangladesh (2012-2050)

Year	Population of Non Linear Model
2012	15.0425
2013	15.2501
2014	15.4611
2015	15.6761
2016	15.8960
2017	16.122
2018	16.3556
2019	16.5986
2020	16.8531
2021	17.1215
2022	17.4065
2023	17.7111
2024	18.0386
2025	18.3921
2026	18.7745
2027	19.1885
2028	19.6366
2029	20.1211
2030	20.6441
2031	21.2075
2032	21.8130
2033	22.4621
2034	23.1561
2035	23.8961
2036	24.6830

Table 8.1	
Year	**Population of Non Linear Model**
2037	25.5175
2038	26.4001
2039	27.3311
2040	28.3106
2041	29.3385
2042	30.4145
2043	31.5381
2044	32.7086
2045	33.9251
2046	35.1865
2047	36.4915
2048	37.8386
2049	39.2261
2050	40.6521

(Population in million)

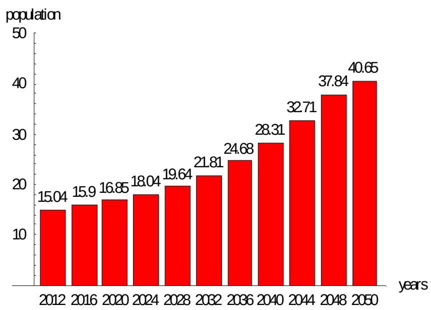

Figure 8.2: Logistic Population Growth Equation Used to Fit the Census Data for the Population of Bangladesh (In Millions)

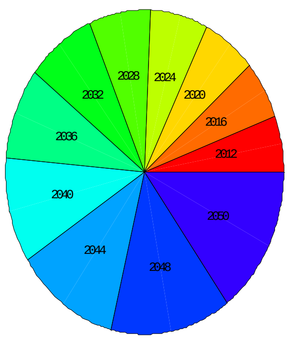

Figure 8.3: Logistic Population Growth Equation Used to Fit the Census Data for the Population Bangladesh (In Million) by Pie Diagram

9.0 CONCLUSIONS

We have discussed in this paper of some basic concepts, dynamical behavior and numerical solution of the logistic mathematical model based on population dynamics easily. We analyzed the data of Belgium, USA and France. Then we used the same method to project population for the aforementioned period of Bangladesh. We observed that our data for the same period of time fitted with a good agreement with the census data. In this study a mathematical analysis of the future population of Bangladesh is carried out based on an ordinary differential equation model which is called logistic model. MATLAB program based on an algorithm of Runge-Kutta scheme is developed for direct calculation of future populations. Then we establish a non-linear model that gave future population in Bangladesh at the time from 1971 to 2011 and 2012 to 2050. The realistic mathematical logistic model is useful in many applications from experimental instruments to rigorous mathematical analysis techniques. Finally, we have noticed that the population of Bangladesh is increasing without bound.

REFERENCES

1. Verhulst, Pierre-François (1845). "Recherches mathématiques sur la loi d'accroissement de la population" [Mathematical Researches into the Law of Population Growth Increase]. Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles 18: 1–42. Retrieved 2013-02-18.
2. G. McKendricka; M. Kesava Paia1 (January 1912). "XLV.—The Rate of Multiplication of Micro-organisms: A Mathematical Study". Proceedings of the Royal Society of Edinburgh 31: 649–653. doi:10.1017/S0370164600025426
3. Pearl, R., & Reed, L. J. (1920). On the Rate of Growth of the Population of the United States Since 1790 and its Mathematical Representation. Proceedings of the National Academy of Sciences, 6 (6), 275-288.
4. Yukalov, V. I.; Yukalova, E. P.; Sornette, D. (2009). "Punctuated evolution due to delayed carrying capacity". Physica D: Nonlinear Phenomena 238 (17): 1752. doi:10.1016/j.physd.2009.05.011
5. Murray, J. D. (2002). Mathematical Biology: An Introduction, Volume 1 (Third Edition). New York, NY: Springer.
6. Kabir, M. and Chowdhury, A. A.. Population Growth and Food Production in Bangladesh", Rural Demography, IX (1 & 2) (1982) 25-56.